数理解析研究所講究録 1248 巻 2002 年 69-74

Nonwandering sets of the powers of maps of trees

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Abstract. Let X be a tree. Then there exists a positive integer $m_X \geq 2$ such that for each a continuous map $f : X \to X$, the nonwandering set of f is equal to the nonwandering set of f^m for every $m \geq 1$ such that the greatest common divisor of m and $m_X!$ is equal to one.

1 Introduction.

A *continuum* is a nonempty, compact, connected, metric space. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are disjoint or intersect only in one or both of their end points. From now on, X denotes a *tree* by which we mean a graph which contains no simple closed curve.

Let f be a continuous map from a continuum X to itself. We denote the n-fold composition f^n of f with itself by $f \circ \cdots \circ f$. Let f^0 denote the identity map. A point $x \in X$ is a *periodic point of period* $n \geq 1$ *for* f if $f^n(x) = x$. A point $x \in X$ is an *eventually periodic point of period* n *for* f if there exists $m \geq 0$ such that $f^{n+i}(x)=f^i(x)$ for all $i \geq m$. That is, $f^i(x)$ is a periodic point of period n for $i \geq m$. A point $x \in X$ is *nonwandering point for* f if for any open set U containing x there exist $y \in U$ and $n > 0$ such that $f^n(y) \in U$. This paper investigates the nonwandering sets of the powers (under composition) of a map f.

We denote the set of periodic points for f, eventually periodic points for f, and nonwandering points for f by $\mathrm{P}(f), \mathrm{EP}(f)$ and $\Omega(f)$, respectively. It is known that the closure of $\mathrm{P}(f)$ is contained in $\Omega(f)$, that $\Omega(f)$ is a closed subspace of X and an invariant set of f i.e. $f(\Omega(f)) \subset \Omega(f)$ (See [BC,p.77]). And if X is a tree, by [HK, p.36], $\Omega(f)$ is contained in the closure $\overline{\mathrm{EP}(f)}$ of $\mathrm{EP}(f)$.

It is clear that $\mathrm{P}(f) = \mathrm{P}(f^n)$ and $\Omega(f^n) \subset \Omega(f)$ for all $n \geq 1$. And it is known that $\Omega(f) = \Omega(f^n)$ does not hold in general. In fact, there exists a continuous map $f : [0,1] \to [0,1]$ such that $\Omega(f^{2^n}) \neq \Omega(f^{2^{n-1}})$ for all $n \geq 1$. By [CN, p.9], if X is an interval, then $\Omega(f) = \Omega(f^n)$ for all odd $n \geq 1$. This result is obtained by two steps :
*Step*1 [CN, Theorem 1, p.10]. If $x \in \Omega(f) \setminus \mathrm{EP}(f)$, then $x \in \Omega(f^n)$ for all $n \geq 1$.
*Step*2 [CN, Theorem 3, p.10]. If $x \in \Omega(f) \cap \mathrm{EP}(f)$, then $x \in \Omega(f^m)$ for all odd $m \geq 1$.

First, we shall prove the following theorem.

Theorem 1 *Let f be a continuous map from a tree X to itself. If $x \in \Omega(f) \setminus \mathrm{EP}(f)$, then $x \in \Omega(f^n)$ for every $n \geq 1$.*

Let x be a point of X. Denote the number of all components of $X \setminus \{x\}$ by $\mathrm{Ord}(x, X)$. Let $En(X) = \{x \in X | \mathrm{Ord}(x, X) = 1\}$ be the set of *end points* of X. We see that $m \geq 1$

is odd if and only if the greatest common divisor $(m, 2)$ of m and 2 is equal to one. Since $|En([0,1])| = 2$, for Step 2 we can rewrite that $x \in \Omega(f^m)$ for all $m \geq 1$ with $(m, |En([0,1])|) = 1$. And by [CN, Proposition 1.1], there exist a continuous map f from $[0,1]$ to itself with $x_0 \in \Omega(f)$ such that $x_0 \in \Omega(f^m)$ if and only if $(m, |En([0,1])|) = 1$. For each positive integer $n > 1$, we say a subspace $X_n = \{re^{i\theta} : \theta = 2k\pi/n (k = 0, 1, \cdots, n-1)$ and $0 \leq r \leq 1\}$ of the complex plane an *n-od*.

Example 1. There exist a continuous map f from an n-od X_n to itself with $x_0 \in \Omega(f)$ such that $x_0 \in \Omega(f^m)$ if and only if $(m, n) = 1$.

This shows that for each $n' \leq n$, there exist a continuous map $f_{n'}$ from an n-od X_n to itself and $x_{n'} \in \Omega(f_{n'})$ such that $x_{n'} \in \Omega((f_{n'})^m)$ if and only if $(m, n') = 1$. We notice that $(m, n') = 1$ for all $n' \leq n$ if and only if $(m, n!) = 1$. Let $B(X) = \{x \in X | \mathrm{Ord}(x, X) \geq 3\}$ be the set of *branched points* of X. And we construct a map g from a tree Y_n with $|En(Y_n)| = n$ to itself with $y_0 \in \Omega(g)$ such that $\mathrm{Ord}(b, Y_n) = 3$ for all $b \in B(Y_n)$ and that $y_0 \in \Omega(g^m)$ if and only if $(m, n) = 1$.

Let $Ed(X)$ be the set of all components of $X \setminus B(X)$. Next we shall prove the following:

Theorem 2 *Let f be a continuous map from a tree X to itself and let*

$$m_X = \max\{|En(X)| + |B(X)|, 2|Ed(X)| + |B(X)| - |En(X)| - 1\}.$$

If $x \in \Omega(f) \cap \mathrm{EP}(f)$, then $x \in \Omega(f^m)$ for every $m \geq 1$ with $(m, m_X!) = 1$. In paticularly, if X is an od, then $x \in \Omega(f^m)$ for every $m \geq 1$ with $(m, |En(X)|!) = 1$.

A *dendrite* is a locally connected,uniquly arcwise connected continuum. We see that every tree is a dendrite.

Example 3. There exist a dendrite S which is not a tree, a continuous map $f : S \to S$ and $x_1 \in \Omega(f) \cap \mathrm{EP}(f)$ such that $x_1 \notin \Omega(f^k)$ for all $k > 1$.

By Example 3, we see trees in Theorem 2 can not be change into dendrites. From Theorem 1 and 2, we have the following main theorem.

Theorem 3 *Let f be a continuous map from a tree X to itself and let*

$$m_X = \max\{|En(X)| + |B(X)|, 2|Ed(X)| + |B(X)| - |En(X)| - 1\}.$$

Then $\Omega(f) = \Omega(f^m)$ for every $m \geq 1$ with $(m, m_X!) = 1$. In paticularly, if X is an od, then $\Omega(f) = \Omega(f^m)$ for every $m \geq 1$ with $(m, |En(X)|!) = 1$.

2 Examples.

Example 1. Let $n \geq 2$ be an integer. We construct a continuous map f from an n-od X_n to itself with $x_0 \in \Omega(f)$ such that $x_0 \in \Omega(f^m)$ if and only if $(m, n) = 1$. Denote $z_k = 1/2e^{2k\pi i/n} (k = 0, 1, \cdots, n-1)$, $x_0 = 3/4e^{2\pi i/n}$, $y_0 = 7/12e^{2\pi i}$, and $x_1 = 2/3e^{2\pi i}$. Define

$$\begin{array}{ll} f(re^{2k\pi i/n}) = re^{2(k+1)\pi i/n} & \text{if } 0 \leq r \leq 1/2 \text{ or } k = 1, \cdots, n-2, \\ f(re^{2(n-1)\pi i/n}) = re^{2\pi i} & \text{if } 0 \leq r \leq 3/4, \text{ and} \\ f(re^{2(n-1)\pi i/n}) = x_0 & \text{if } 3/4 \leq r \leq 1. \end{array}$$

And define

$$
\begin{aligned}
&f(tz_0 + (1-t)y_0) = tz_1 + (1-t)e^{2\pi i/n},\\
&f(ty_0 + (1-t)x_1) = te^{2\pi i/n},\\
&f(tx_1 + (1-t)x_0) = (1-t)z_0, \text{and}\\
&f(tx_0 + (1-t)e^{2\pi i}) = tz_0 + (1-t)x_0 \quad (0 \le t \le 1).
\end{aligned}
$$

Set $e'_k = \{re^{2k\pi i/n} : 3/4 \le r \le 1\}(k = 0, 1, \cdots, n-1)$ and $z'_0 = 15/24 e^{2\pi i}$ with $f(z'_0) = z_1$. We note $f^{-1}(x_0) = \{e^{2\pi i}\} \cup e'_{n-1}$ and $f(X_n) = \overline{X_n \setminus e'_0}$. We see that $f^{-n+1}(x_0) = e'_1$ for $n \ge 3$ and that there exist $y_1 \in (z_0, y_0)$ and $z'_1 \in (z'_0, y_0)$ such that $f^{-n}(x_0) = [y_1, z'_1]$. Set $x'_1 = 3/4e^{2\pi i/n} = f(y_1) = f(z'_1)$. We have $x'_2, x''_2 \in (z_1, x'_1)$ such that $f^{n-1}(x'_2) = y_1, f^{n-1}(x''_2) = z'_1$ and $f^{-n+1}([y_1, z'_1]) = [x'_2, x''_2]$. Therefore, there exist $y_2, y'_2 \in (z_0, y_1)$ and $z'_2, z''_2 \in (z'_0, z'_1)$ such that $f(y_2) = f(z'_2) = x'_2, f(y'_2) = f(z''_2) = x''_2$, and $f^{-n}([y_1, z'_1]) = [y_2, y'_2] \cup [z'_2, z''_2]$. After all, we have $f^{-2n}(x_0) = [y_2, y'_2] \cup [z'_2, z''_2]$. Inductively, for each $m \ge 2$, there exist $y_m, y'_m \in (z_0, y_{m-1})$ and $z'_m, z''_m \in (z'_0, z'_{m-1})$ such that $f^{-mn}(x_0) = [y_m, y'_m] \cup [z'_m, z''_m]$. And we see that $\lim_{m\to\infty} y_m = z_0$, $\lim_{m\to\infty} z'_m = z'_0$, $U \cap \bigcup_{k=1}^{\infty} f^{-k}(x_0) = U \cap \bigcup_{k=1}^{\infty} f^{-kn}(x_0) \subset \bigcup_{m\ge1}[y_m, y'_m]$ for each small connected neighborhood U of z_0 and that for all $m' > 0$,

$$
\begin{aligned}
& f^{(m'+m+1)n}([z_0, y_m])\\
= \;& f^{(m'+1)n}(f^{mn}([z_0, y_m]))\\
= \;& f^{(m'+1)n}([z_0, x_0])\\
= \;& f^{m'n}([0, x] \cup [0, z_{n-1}])\\
= \;& f^{(m'-1)n}([0, x_0] \cup [0, z_{n-1}])\\
& \cdots\\
= \;& f^{n}([0, x_0] \cup [0, z_{n-1}])\\
= \;& [0, x_0] \cup [0, z_{n-1}].
\end{aligned}
$$

This shows that for each small connected neighborhood V of x_0, there exists $m > 0$ such that $x_0 \in f^k(V)$ if and only if $k = (m + m')n + 1$ for some $m' \ge 0$. We conclude that $x_0 \in \Omega(f^m)$ if and only if $(m, n) = 1$.

This shows that for each $n' \le n$, there exist a continuous map $f_{n'}$ from an n-od X_n to itself and $x_{n'} \in \Omega(f_{n'})$ such that $x_{n'} \in \Omega((f_{n'})^m)$ if and only if $(m, n') = 1$.

Example 2. Let $Y_n = \{(x, y) : 0 \le x \le n-1 \text{ if } y = 0, \text{ or } 0 \le y \le 6 \text{ if } x = 0, 1, \cdots n-1\}$ $(n \ge 3)$. We construct a continuous map g from Y_n to itself with $y_0 \in \Omega(g)$ such that $y_0 \in \Omega(g^m)$ if and only if $(m, n) = 1$.

Let $y_0 = (0, 5)$ and $z_0 = (0, 2)$. Define

$$
\begin{array}{ll}
g((x,0)) = (x,0) & \text{if } 0 \le x \le n-1,\\
g((x,y)) = (x+1,y) & \text{if } 1 \le y \le 6 \text{ and } x = 1, 2, \cdots, n-2,\\
g([(x,0),(x,1)]) = [(x,0),(x+1,1)] & \text{if } x = 0, 1, \cdots, n-2,\\
g((n-1,y)) = (0,y) & \text{if } 1 \le y \le 5,\\
g([(n-1,0),(n-1,1)]) = [(n-1,0),(0,1)], & \\
g((n-1,y)) = y_0 & \text{if } 5 \le y \le 6,\\
g((0,1+t)) = (1,1+t), g((0,2+t)) = (1,2+4t), & \\
g((0,3+t)) = (1,6-6t), g((0,5+t)) = (0,2+3t) & \text{for } 0 \le t \le 1, \text{ and}\\
g([(0,4),(0,5)]) = [(1,0),z_0]. &
\end{array}
$$

As the proof of Example 1, we can show that for each small connected neighborhood V of y_0, there exists $m > 0$ such that $y_0 \in g^k(V)$ if and only if $k = (m + m')n + 1$ for some $m' \ge 0$. We conclude that $y_0 \in \Omega(g^m)$ if and only if $(m, n) = 1$.

This shows that for each $n' \leq n$, there exist a continuous map $g_{n'}$ from Y_n to itself and $y_0 \in \Omega(g_{n'})$ such that $y_0 \in \Omega((g_{n'})^m)$ if and only if $(m, n') = 1$.

Example 3. Let S be a subspace $\{re^{i\theta} : n = 1, 2, \cdots, \theta = 2\pi/n \text{ and } 0 \leq r \leq 1/n\}$ of the complex plane and $x_1 = 1/2e^{2\pi i}$. We construct a continuous map $f : S \to S$ such that $x_1 \in \Omega(f) \cap \mathrm{EP}(f)$ and $x_1 \notin \Omega(f^k)$ for all $k > 1$.

Let $p_2, p_3, \cdots$ be a sequence of primes with $1 < p_n < p_{n+1}$ $(n \geq 2)$. Denote $I_n = \{re^{2\pi i/n} : 0 \leq r \leq 1/n\} \subset S$ and $J_n = \{re^{2\pi i} : 1/2 + 1/2n \leq r \leq 1/2 + 1/2(n-1)\}$ for each $n = 2, 3, \cdots$.

Define $f(\{re^{2\pi i} : 0 \leq r \leq 1/2 \text{ or } r = 1/2 + 1/2n \text{ for each } n = 2, 3, \cdots\}) = \{0\}$, $f(re^{2\pi i/n}) = rn/(n-1)e^{2\pi i/(n-1)}$ for each $n > 2$, $f(re^{\pi i}) = r/2e^{2\pi i}$ for $0 \leq r \leq 1$ and $f(J_n) = I_{p_n}$ for each $n = 2, 3, \cdots$.

Let U be a small connected neighborhood of x_1 in S. We have that $x_1 \in f^k(U)$ if and only if $k = p_n$ for some $n \geq 2$. We conclude that $x_1 \in \Omega(f)$ and $x_1 \notin \Omega(f^k)$ for all $k > 1$.

3 The proof of Theorem 2.

We will omit the proof of Theorem 1.

Let f be a continuous map from a tree X to itself and let z_0 be a periodic point of f with period $n \geq 1$. Set $z_i = f^i(z_0)$. Put $X \setminus \{z_i\} = X_i(0) \cup X_i(1) \cup \cdots \cup X_i(r_i)$ for some $r_i \geq 0$, where each $X_i(j)$ is a component of $X \setminus \{z_i\}$. If V is a neighborhood of z_0 in $\overline{X_0(0)}$, we say V a *0-neighborhood* of z_0. Set $W(z_0, f^n, X_0(0)) = \{x \in X |$ for any 0-neighborhood V of z_0, $x \in f^{nk}(V)$ for some $k > 0$ $\}$ and $W_i = f^i(W(z_0, f^n, X_0(0)))$.

Let $x \in \Omega(f) \cap \mathrm{EP}(f) \setminus \overline{\mathrm{P}(f)}$, let q be the least positive integer such that $f^q(x) \in \mathrm{P}(f)$ and let $f^q(x)$ have period n. Since $x \in \Omega(f)$, we have $x = f^{n_k}(x_k)$, where $x_k \to x$ $(n_k \to \infty)$. For some $p \geq q$ we have $n_k \equiv p \pmod{n}$ for infinitely many k. Then $f^p(x_k) \to f^p(x) = z_0$ $(n_k \to \infty)$. Moreover we may suppose that all $f^p(x_k) \in X_0(0)$. Put $W_j = f^j(W(z_0, f^n, X_0(0)))$, so that $x \in W_0$. There exists unique integers s, t with $0 \leq s < n$ and $t > 0$ such that $p + s = tn$. Set $z_i = f^i(z_0)$ and $k_0' = \max\{\ \mathrm{Ord}(z_i, X) : i \geq 0\}$.

Lemma 1 *Let f be a continuous map from a tree X to itself, let $x \in \Omega(f) \cap \mathrm{EP}(f) \setminus \mathrm{P}(f)$, and let $f^s(x)$ be a periodic point of period $n \geq 1$ for some $s \geq 1$. If $k_0 = \min\{\mathrm{Ord}(f^p(x), X) : p \geq s\}$, there exists an integer ℓ with $1 \leq \ell \leq k_0$ such that then for each neighborhood G of x we have a positive integer $N(G)$ such that $x \in f^{\ell jn + N(G)}(G)$ for all $j \geq 0$.*

Lemma 2 *Suppose that $x \in W_s$. Then there exists an integer ℓ with $1 \leq \ell \leq k_0'$ such that $x \in \Omega(f^m)$ for every $m \geq 1$ with $(m, \ell) = 1$.*

Lemma 3 *Let $k_0' = \max\{\mathrm{Ord}(z_i, X) : 0 \leq i < n\}$. If $k_1 = 2$, then there exists an integer ℓ with $1 \leq \ell \leq k_0'$ such that $x \in \Omega(f^m)$ for every $m \geq 1$ with $(m, 2\ell) = 1$.*

Lemma 4 *Suppose that $x \notin W_s$ (i.e. $W_0 \neq W_s$). Then $2 \leq k_1 \leq |En(X)| + |B(X)|$. In paticularly, if X is an od, then $2 \leq k_1 \leq |En(X)|$.*

Lemma 5 *Suppose that $x \notin W_s$ (i.e. $W_0 \neq W_s$) and that $k_1 \geq 3$. Then*

(1) *Then $s \leq 2|Ed(X)| + |B(X)| - |En(X)| - 1$.*

(2) *Paritcularly, if X is an od, then $s \leq |En(X)| - 1$.*

If the lemmas above can be proved, Theorem 2 can be proved as follows.

Proof of Theorem 2. We may assume that $x \in \Omega(f) \cap \mathrm{EP}(f) \setminus \overline{\mathrm{P}(f)}$. Let q be the least positive integer such that $f^q(x) \in \mathrm{P}(f)$ and let $f^q(x)$ have period n. And let $m \geq 1$ with $(m, m_X!) = 1$.

Suppose that $x \in W_s$. Since $k'_0 \leq |En(X)| \leq m_X$, by Lemma 2, we have $x \in \Omega(f^m)$.

Suppose that $x \notin W_s$. Since $k'_0 \leq m_X$, by Lemma 3, we may assume that $k_1 \geq 3$. By Lemma 1, there exists an integer ℓ with $1 \leq \ell \leq k_0$ such that for each neighborhood G of x we have a positive integer N such that $x \in f^{\ell jn+N}(G)$ for all $j \geq 0$. There exists a positive integer k' such that $k'n = k_1 s$. Since $k_0 \leq m_X$, from Lemma 4 and 5, we see that $(m, \ell n) = 1$. There exist two integers p, q such that $-p\ell n + qm = N$. We conclude that $x \in f^{qm}(G)$ and that $x \in \Omega(f^m)$.

4 Questions.

Let X be a tree. Set $n_X = \min\{n : \Omega(f) = \Omega(f^m)$ for all continuous map $f : X \to X$ and all $m \geq 1$ with $(m, n!) = 1\} \geq |En(X)|$ (by Example 2).

Question 1. Do there exists a tree X_0 with $n_{X_0} < m_{X_0}$?

Question 2. Does there exist a continous map $f : X \to X$ such that for each $\ell \leq n_X$, we have $x_\ell \in \Omega(f)$ such that $x_\ell \in \Omega(f^m)$ if and only if $(m, \ell) = 1$?

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